

Properties of low variability periods in financial time series

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Abstract

Properties of low-variability periods in the time series are analysed. The theoretical approach is used to show the relationship between the multi-scaling of low-variability periods and multi-affinity of the time series. It is shown that this technically simple method is capable of revealing more details about time series than the traditional multi-affine analysis. We have applied this scaling analysis to financial time series: a number of daily currency and stock index time series. The results show a good scaling behaviour for different model parameters. The analysis of high-frequency USD-EUR exchange rate data confirmed the theoretical expectations.

Key words: Econophysics, multiscaling, multifractality, time series analysis, scale-invariance

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1 Introduction

Perhaps the only thing which is more complicated and unpredictable than a human mind, is the collective human mind. This collective human mind is the driving force of financial market fluctuations. The intrinsic complexity of the market dynamics forms the basis for the exponential growth of the econophysics, since the publication of seminal papers of Stanley et al [1,2,3]. As in every new scientific discipline, the Econophysics is developing rapidly in various directions. Research in those various fields has contributed several stylized statistical properties of asset returns (which, in principle, may cease to be valid at a certain moment of time, because the collective human mind is

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capable of learning), c.f. [4,5,6]. Most advancements in econophysics are owing to the concepts of scale-invariance, intermittency and heteroscedasticity. So, viable approximations for the market movements have been either based on the continuous time random walk (CTRW) model [7,8,9,10,11,12], or derived from it, c.f. the model of fractional Brownian motion (fBm) in multifractal trading time (fBm-mtt) [13,14,15]. There is strong empirical evidence that fluctuations in financial markets possess multifractal statistics [16,17,18,19,20]. Multifractality has been found to be inherent both to the currency markets [16,18,19], and to the stock exchange markets [20]. The time series analysis is usually based on the closing prices of two consecutive trading days, but higher frequency data have been also used [21].

The focus of this paper is placed on the multi-scaling properties of currency and stock exchange time series. So, we refrain ourselves from disussing the advancements in understanding the other aspects of financial time series.

It has been recently shown that in the case of intermittent time series, the scaling behaviour of low-variability periods can provide additional information, as compared to a multifractal analysis [22]. Preliminary study of the time series of stock indices and currency rates indicated that similarly to the heart rate variability data, the financial time series are also characterized by a multi-scaling behaviour of low-variability (i.e. “silent” or “calm”) periods [23]. Such a behaviour has been independently verified by Kaizoji et al [24].

In this paper, analytic approach is used to derive the relationship between the multi-scaling of low-variability periods, and multi-affinity of the time series. Besides, a detailed analysis of the statistics of low variability periods is performed for various daily stock indices and daily currency rates. The study of the high-frequency data of the USDEUR exchange rate indicates that at the time-scale of one day, there is a cross-over between two scaling laws for the low-variability periods.

2 Theoretical approach to low-variability periods

Most modern studies of the variability of intermittent time series are based on (or derived from / related to) the model of multi-affine fractional Brownian motion (mafBm), cf. [4,13,25,26,27]. This is not surprising, because multifractal behaviour is believed to be the most universal case of scale-invariance, c.f. [28]. Note that while CTRW and fBm-mtt models are somewhat more detailed than a data description by multifractal formalism (by introducing the concept of waiting- or trading time), the time series generated by both these models can be also characterized by the multi-affine spectra and lead to the same scaling of low-variability periods as the model of mafBm.

2.1 Multi-affine fractional Brownian motion

One of the disadvantages of the multi-affine analysis is neglecting the long-term correlations in the dynamics of high-frequency variability. This claim is motivated as follows. Consider a multi-affine time series $x(t)$, with a lower cut-off scale τ_0 . Then, each point of the time series t_0 is characterized by its own Hurst exponent h (referred to as the Lipschitz-Hölder exponent); this exponent describes the local scaling of fluctuations,

$$|x(t) - x(t_0)| \sim |t - t_0|^h, \quad |t - t_0| \gg \tau_0. \quad (1)$$

Further, the distribution of points of certain values of h is self-similar, and is described by a fractal dimension $f(h)$. So, according to the mafBm model, we study the scaling of increments $|x(t) - x(t_0)|$ at $|t - t_0| \gg \tau_0$, without specific attention to the values of them at $|t - t_0| \approx \tau_0$. In order to shed light into this phenomenon, the method of scaling analysis of low-variability periods has been devised [22].

On the other hand, a low-variability period of length l_i is defined as such a continuous time interval $T_i = [t_i, t_i + l_i]$ ($i = 1, 2, \dots$) that (a)

$$|x(t) - \langle x(t) \rangle_\tau| \leq \delta \quad \text{for } t \in T_i \quad (2)$$

where δ is a threshold parameter and angular braces denote sliding average over a window of width $\tau > \tau_0$ (in principle, the window width can be arbitrarily large; however, the highest time resolution of the method and widest scaling range is achieved when it is as small as possible, i.e. just few cut-off scales τ_0); (b) each period has maximal possible length, implying that decreasing t_i or increasing l_i would lead to violation of Eq. 2. We speak about multiscaling behaviour, if the cumulative distribution function of the low-variability periods (the number of periods with $l_i \geq n$) scales as

$$R(n) = R_0 n^{-\alpha(\delta, \tau)}, \quad (3)$$

where $\alpha(\delta, \tau)$ is a scaling exponent and R_0 is a constant.

It should be noted that the scaling exponent $\alpha(\delta, \tau)$ can be considered as the extension of the concept of multi-affine spectrum $f(h)$ to the highest possible time resolution. This is because of two circumstances. First, the effective time resolution of the multi-affine description by Eq. 1 is limited by the required scaling range $|t - t_0| \gg \tau_0$; meanwhile, the effective time resolution of the description based on low-variability periods is the cut-off scale τ_0 . Second, in

the case of infinite time resolution, there is one-to-one correspondence between the spectra $f(h)$ and $\alpha(\delta, \tau)$, see below.

The relationship between $f(h)$ and $\alpha(\delta, \tau)$ spectra can be derived as follows. If a time moment t is characterized by a Lipschitz-Hölder exponent h , the mismatch between the local average and the instant value is of the same order of magnitude as the variability amplitude of $x(t)$ over the time window of width τ :

$$|x(t) - \langle x(t) \rangle_\tau| \approx \tau^h. \quad (4)$$

Comparing Eqns 2 and 4, we conclude that for a period T_i , all the points are characterized by $h \leq \log_\tau \delta$. Similarly, the edges $t_i - \tau_0$ and $t_i + l_i + \tau_0$ of the period T_i are characterized by $h > \log_\tau \delta$. Therefore, the low-variability periods are the contiguous pieces which remain from the entire time-axis after the removal of the high-variability points with $h > \log_\tau \delta$. These high-variability points form a sparse Cantor-dust-like set, assuming that $\log_\tau \delta > h_0$, where h_0 corresponds to the global maximum of the spectrum $f(h)$ [i.e. $f(h_0)=1$]. Indeed, $f(h)$ is the fractal dimension of the set of points described by the exponent h . So, the fractal dimension of the set of points described by exponents $h > \log_\tau \delta$ is found as

$$d = \sup_{h > \log_\tau \delta} f(h) = \begin{cases} 1, & \text{if } \log_\tau \delta \leq h_0 \\ f(\log_\tau \delta), & \text{if } \log_\tau \delta > h_0 \end{cases}. \quad (5)$$

If an interval is divided into pieces by a Cantor dust of dimension d , the cumulative length-distribution exponent of the pieces is also d (because the number of pieces larger than n is estimated as the number of boxes of size n required to cover the Cantor dust, according to the box-counting method). So, we have correspondence

$$\alpha(\delta, \tau) = f(\log_\tau \delta) \quad \text{if } \log_\tau \delta > h_0. \quad (6)$$

Note that in the case of simple fBm (which is studied in more details in Ref. [29]), there is no scaling of low-variability periods with lengths $l > \tau$. Indeed, for simple fBm, $f(h) = 1$, if $h = H$ [otherwise, $f(h)$ is not defined]. So, the high-variability points either are absent (i.e. there is a single low-variability period), or they are populated quasi-homogeneously with fractal dimension $d = 1$ (then, all the low-variability periods are shorter than or of the order of the window width τ).

2.2 Multi-affine trading time and discussions

The approach used for mafBm can be also applied to the fBm-mtt model with $x(t) = F(\mathcal{T}(t))$, where F is a fractional Brownian function of its argument (described by the Hurst exponent H) and $\mathcal{T}(t)$ is a multi-affine function of time. Indeed, in that case Eq. 4 is substituted by

$$|x(t) - \langle x(t) \rangle_\tau| \approx (\Delta_\tau T)^H \approx \tau^{hH}. \quad (7)$$

Here, $\Delta_\tau T \approx \tau^h$ is the increment of the trading time $\mathcal{T}(t)$ and h is the local Lipschitz-Hölder exponent for the function $\mathcal{T}(t)$. Accordingly, Eq. 6 is substituted by

$$\alpha(\delta, \tau) = f(H^{-1} \log_\tau \delta) \quad \text{if} \quad \log_\tau \delta > H h_0. \quad (8)$$

So, both in the case of the mafBm and fBm-mtt models, the scaling described by multifractal spectrum $f(h)$ with $h > h_0$ can be also described by the scaling of low-variability periods. The approach to describing the Lipschitz-Hölder exponent range $h < h_0$ is straightforward: instead of low-variability periods, high variability periods have to be used. Then, Eqns. 6 and 8 will remain valid, except that the inequalities will be opposite; details of this aspect will be analysed elsewhere. Now, we are left with two questions.

First, the multifractal dimension $f(h)$ cannot exceed the topological dimension one. Is it possible to have the scaling exponent of low-variability periods larger than one? The answer is yes (examples are provided below). More specifically, equality $\alpha > 1$ assumes that the most part of the aggregated time T_{agg} of the low-variability periods corresponds to shortest periods (because $T_{\text{agg}} = \int n \cdot dR(n) = R_0 \int n^{-\alpha-1} dn$; here, the integral vanishes at large lengths n). This assumes that the fractal dimension of the high variability periods is one, i.e. $\log_\tau \delta < h_0$. In that case, the low-variability moments form a Cantor-dust like set of points. For real time series, the time resolution is always finite; then, in the regions of high “dust density”, the “dust particles” may overlap and form relatively long (much longer than the time resolution, but much shorter than the total time interval) low-variability periods. Such a clustering of small- h -points is not described by the multifractal spectrum $f(h)$.

Second, the multifractal spectrum $f(h)$ is a one-parameter-curve; meanwhile, $\alpha(\delta, \tau)$ is a two-parameter-curve, which is effectively reduced to one degree of freedom by Eq. 6 [or 8]. Is it possible to have a such a class of scale-invariant time series, which is more generic than the class of multi-affine functions, so that the exponent α can be arbitrary function of two independent variables δ and τ . This question is left open for further studies.

To conclude, the scaling analysis of low variability periods provides a simple and superior (in the sense of time resolution and applicability to the parameter range with $\alpha > 1$) alternative to the traditional multi-affine analysis of the time series. In particular, Eq. 5 provides a way to check the validity of the assumption of multi-affinity, which requires $\alpha(\delta, \tau) \equiv \alpha(\log_\tau \delta)$. If there is no need for such verification, the route is even simpler, e.g. the analysis can be performed at a fixed window length τ . All this does not mean that studying the scaling of low-variability periods is always better than the multi-fractal formalism. Instead, the two methods should be considered as complementing each other, because the accuracy and applicability of both methods are sensitive with respect to the width of the scaling range and specifics of the scaling behaviour.

3 Properties of low-variability periods in financial time series

Now we apply the above developed theory to financial time series. The specifics of long-term stock-market time series is that prices can vary by orders of magnitude. Therefore, it makes sense to work with a logarithmic scale, and substitute the definition of low variability (Eq. 2) as follows:

$$|1 - P(t)/\langle P(t) \rangle_\tau| \leq \delta \text{ for } t \in T_i. \quad (9)$$

Here, unlike in the case of Eq. 2, it is assumed that average is taken not over a time window centered around the current moment of time t , but instead, the large- t -edge of the window coincides with t (because for practical applications in real time, the future is not known):

$$\langle P(t) \rangle_\tau = \frac{1}{\tau} \sum_{k=0}^{\tau-1} P(t-k). \quad (10)$$

So, a low-variability period is the period between two consecutive price movements that are exceeding the given threshold δ . The length of a low variability period is measured in the same units as time t — in the units of the sampling period of the market data. The local average in Eqn 9 gives us the second input parameter: τ that could be interpreted as a degree of “locality”. With these two input variables we maintain the easy interpretation of the results: threshold value δ is simply the percentage change of asset at time t compared against the average price of that asset in preceding period with length of τ time units. Further we measure the lengths of low-variability periods and we count them. Then we define a cumulative distribution function $R(n)$ that represents the amounts of periods where low-variability lasted at least n time units. Finally, we fit the data against the scaling law given by Eq. 3.

Following analysis will give answers to two essential questions:

- (1) Do the low variability periods obey power law and if so, what is the scaling exponent?
- (2) Assuming that there is a power law, how does the scaling exponent depend on the parameters δ and τ ?

3.1 Determination of Power law

We used the following data in our analysis:

Table 1

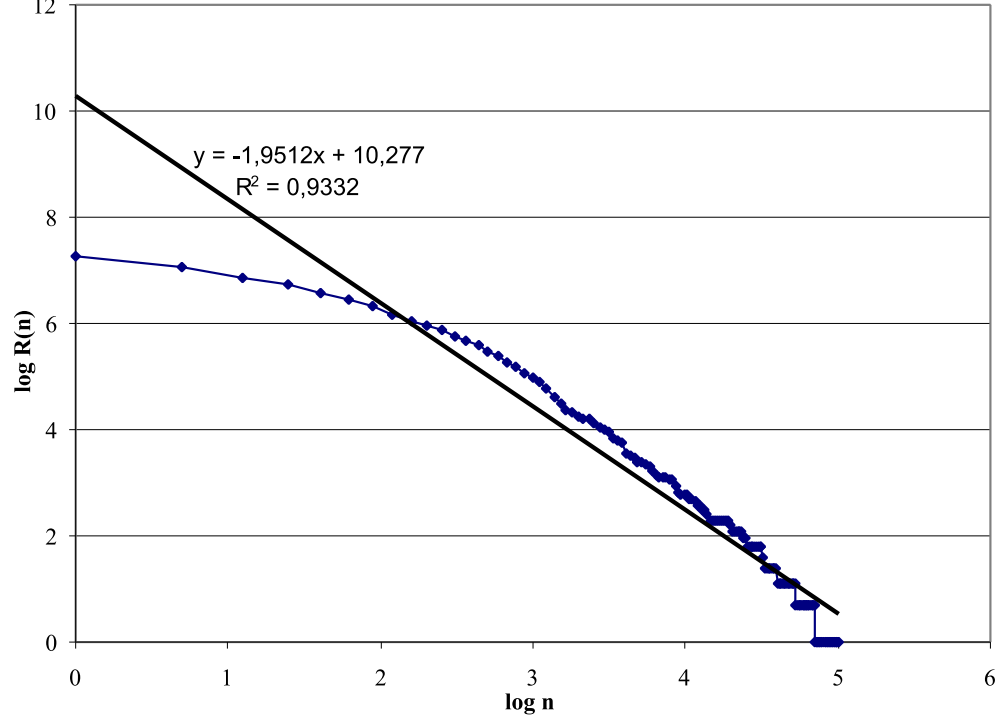
Data used in the analysis				
Abbrev	Description	Calendar period	Frequency	# of data
EUR	EUR / USD exchange rate	01/01/2001 – 31/12/2003	1. min	1106031
AUD	AUD / USD exchange rate	13/12/1983 – 20/04/2004	Daily	5261
CAD	CAD / USD exchange rate	04/01/1971 – 20/04/2004	Daily	8491
DEM*	DEM / USD exchange rate	04/01/1971 – 20/04/2004	Daily	8490
FRF*	FRF / USD exchange rate	04/01/1971 – 20/04/2004	Daily	8491
GBP	GBP / USD exchange rate	04/01/1971 – 20/04/2004	Daily	8476
JPY	JPY / USD exchange rate	04/01/1971 – 20/04/2004	Daily	8485
DAX	German equity index - DAX	01/10/1959 – 20/04/2004	Daily	11169
UKX	UK equity index - FTSE100	03/01/1984 – 20/04/2004	Daily	5134
NKY	Japanese equity index - Nikkei 225	05/01/1970 – 20/04/2004	Daily	8492
EAFE	MSCI Europe Austrasia Far East index	31/12/1969 – 20/04/2004	Daily	8552
World	MSCI World index	31/12/1969 – 20/04/2004	Daily	8553
SPX	Standard & Poors' 500	30/12/1927 – 20/04/2004	Daily	16767
DJIA	Dow Jones Industrial average	03/01/1900 – 20/04/2004	Daily	26206

*Remark: from January, 1st 1999 German Mark and French Franc are fixed against Euro.

In our analysis, the scaling exponent was found by using least-squared fit for the whole data series. For visualizations, the dependence $R(n)$ is plotted in log-log scale. Then, the scaling exponent α is equal to slope of the fit-line. Financial time series are single realizations of intermittently fluctuating nonstationary time series, which makes calculation of exact error estimation of the scaling exponent impossible. However, rough estimates of the uncertainties have been obtained as follows. The least-squares fitted trend-line was found as described above, except that the slope α was not optimized, i.e. it was considered as a fixed parameter. Further, the sum of squared residuals $r(\alpha)$ was calculated as a function of α . The error estimate was found as $e = (\alpha' - \alpha)$, where α is the least-squares fitted value of the slope, and α' satisfies the condition $r(\alpha') = 2r(\alpha)$. The technique used for obtaining the scaling exponent α is illustrated in Fig 1.

Above-mentioned procedure was carried out for all of the time series and

Fig. 1. Determination of the scaling exponent α for SPX time series under the following conditions: $\delta=1.6\%$ and $\tau=10$ days



with different thresholds and window widths. The quality of the data fit was measured by the correlation coefficient R^2 . Total 1004 measurements were carried out with all the daily time series. In Fig 2, the validity of power law hypothesis is demonstrated. Histogram shows that in most cases, the power law provided a good data fit. Calculated values of α and error estimations for currencies and equity indices for $\tau = 10$ days are presented in tables 2 and 3 respectively.

It is important to outline two effects:

- There is reverse relationship between the correlation coefficient and the threshold parameter δ . The reason is that the amount of large movements is small and not representative for statistical analysis.
- There is also reverse relationship between the correlation coefficient and the window width τ , due to similar reasons (longer averaging smoothes time series and large movements become more frequent).

Due to these effects there is no sense to measure the scaling exponents beyond certain parameter range, e.g. for $\tau = 2$ days and $\delta > 2\%$.

Fig. 2. Histogram of R-squared coefficients based on the regression analysis of determining scaling exponent α

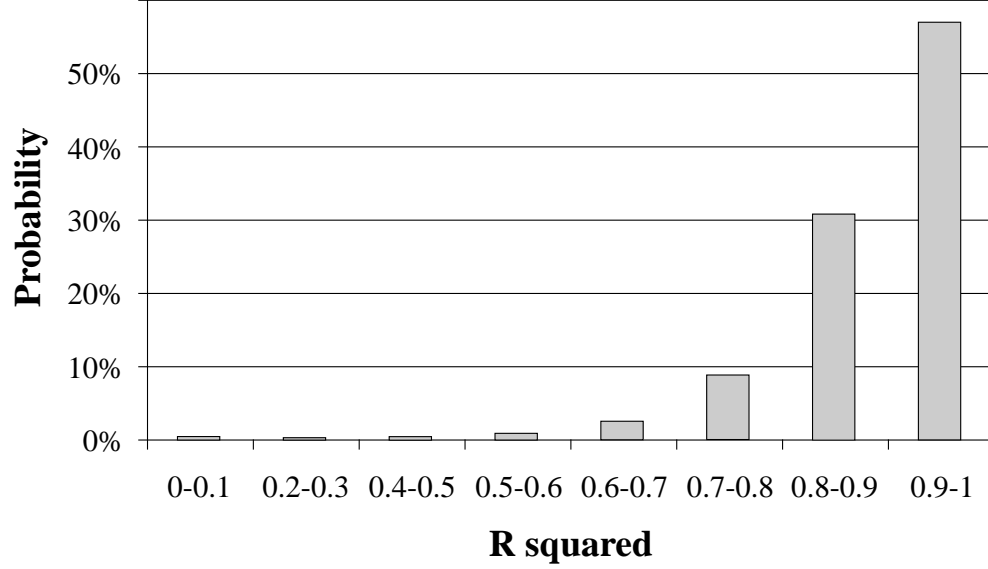


Table 2

Scaling exponents for currency time series with $\tau=10$ days

Threshold $\delta(\%)$	CAD	AUD	GBP	FRF	DEM	JPY
0.10	2.33 ± 0.25	3.41 ± 0.47	1.38 ± 0.07	1.38 ± 0.07	1.88 ± 0.12	1.12 ± 0.06
0.25	1.91 ± 0.09	2.60 ± 0.25	1.35 ± 0.04	1.28 ± 0.08	1.70 ± 0.11	1.43 ± 0.05
0.40	1.64 ± 0.11	2.37 ± 0.19	1.51 ± 0.05	1.23 ± 0.05	1.64 ± 0.05	1.36 ± 0.06
0.55	1.55 ± 0.09	2.27 ± 0.24	1.61 ± 0.06	1.35 ± 0.07	1.64 ± 0.07	1.48 ± 0.06
0.70	1.46 ± 0.08	1.95 ± 0.16	1.72 ± 0.08	1.35 ± 0.06	1.69 ± 0.05	1.54 ± 0.05
0.85	1.35 ± 0.09	1.98 ± 0.19	1.80 ± 0.10	1.43 ± 0.06	1.57 ± 0.07	1.58 ± 0.06
1.00	1.31 ± 0.07	1.80 ± 0.18	1.71 ± 0.08	1.46 ± 0.06	1.66 ± 0.08	1.60 ± 0.07
1.15	1.16 ± 0.06	1.67 ± 0.13	1.77 ± 0.09	1.43 ± 0.07	1.72 ± 0.08	1.65 ± 0.09
1.30	1.09 ± 0.06	1.48 ± 0.08	1.77 ± 0.11	1.38 ± 0.06	1.65 ± 0.09	1.59 ± 0.08
1.45	1.06 ± 0.08	1.59 ± 0.09	1.62 ± 0.11	1.43 ± 0.06	1.58 ± 0.07	1.47 ± 0.09
1.60	0.75 ± 0.07	1.52 ± 0.12	1.52 ± 0.10	1.49 ± 0.06	1.53 ± 0.07	1.48 ± 0.10
1.75	0.73 ± 0.07	1.39 ± 0.12	1.44 ± 0.09	1.49 ± 0.08	1.54 ± 0.08	1.47 ± 0.08
1.90	0.57 ± 0.07	1.24 ± 0.13	1.36 ± 0.11	1.51 ± 0.10	1.43 ± 0.06	1.41 ± 0.08
2.05	0.51 ± 0.06	1.27 ± 0.12	1.02 ± 0.07	1.45 ± 0.12	1.46 ± 0.08	1.27 ± 0.06
2.20	0.51 ± 0.06	1.14 ± 0.13	1.14 ± 0.09	1.19 ± 0.12	1.44 ± 0.09	1.24 ± 0.07
2.35	0.51 ± 0.05	1.13 ± 0.06	1.19 ± 0.09	1.32 ± 0.11	1.21 ± 0.07	1.26 ± 0.07
2.50	0.54 ± 0.05	1.05 ± 0.07	1.01 ± 0.06	1.34 ± 0.10	1.10 ± 0.05	1.26 ± 0.07
2.65	0.45 ± 0.04	0.95 ± 0.05	1.11 ± 0.05	1.17 ± 0.08	1.09 ± 0.06	1.24 ± 0.07
2.80	0.40 ± 0.04	0.92 ± 0.04	0.99 ± 0.05	0.97 ± 0.05	0.98 ± 0.04	1.25 ± 0.08
2.95	0.40 ± 0.04	0.79 ± 0.04	0.80 ± 0.04	0.94 ± 0.05	0.95 ± 0.05	1.22 ± 0.08

3.2 Dependence of the scaling exponent on the parameters

Two phenomena can be observed:

- (1) Exponent α and threshold δ tend to be negatively related: larger δ values correspond to lower α values.
- (2) For equities, the dependence of the exponent α on the threshold δ is

Table 3

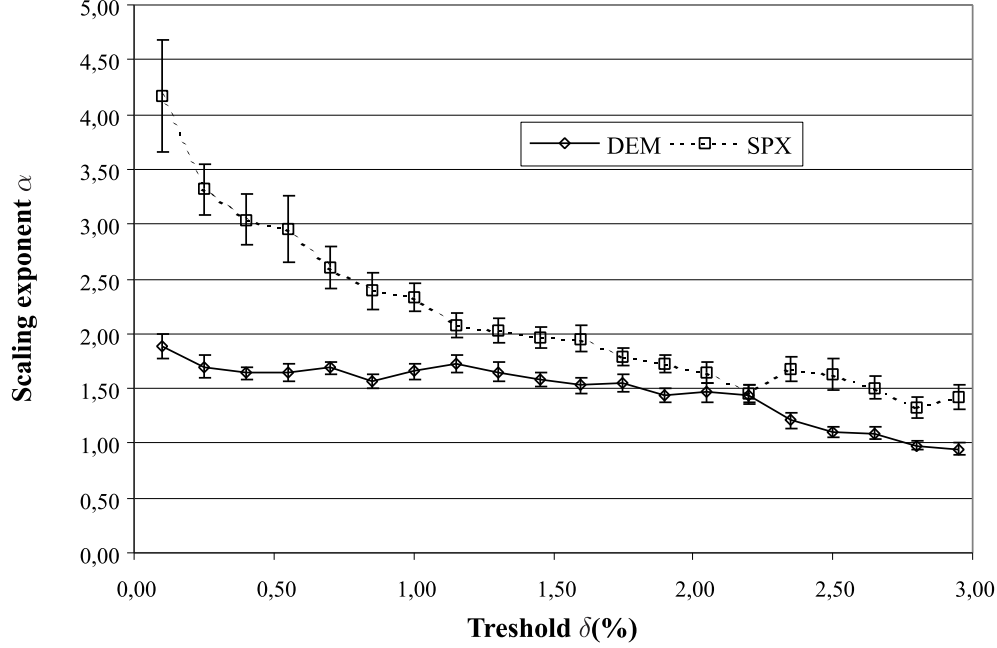
Scaling exponents for equity index time series with $\tau=10$ days

Threshold δ (%)	UKX	DAX	NKY	EAFE	World	SPX	DJIA
0.10	3.87 ± 0.32	4.37 ± 0.38	4.07 ± 0.49	4.27 ± 0.56	3.50 ± 0.36	4.17 ± 0.51	4.03 ± 0.57
0.25	3.02 ± 0.19	3.87 ± 0.43	3.34 ± 0.33	3.16 ± 0.49	2.98 ± 0.32	3.32 ± 0.23	3.75 ± 0.48
0.40	3.09 ± 0.30	3.07 ± 0.31	2.64 ± 0.24	2.58 ± 0.31	2.74 ± 0.37	3.04 ± 0.23	3.09 ± 0.35
0.55	2.93 ± 0.34	3.13 ± 0.35	2.75 ± 0.27	2.43 ± 0.22	2.52 ± 0.26	2.95 ± 0.31	2.84 ± 0.32
0.70	2.54 ± 0.38	2.85 ± 0.29	2.35 ± 0.20	2.43 ± 0.19	2.63 ± 0.28	2.60 ± 0.19	2.72 ± 0.40
0.85	2.44 ± 0.43	2.72 ± 0.24	2.21 ± 0.17	2.36 ± 0.21	2.34 ± 0.20	2.39 ± 0.16	2.74 ± 0.26
1.00	2.38 ± 0.26	2.54 ± 0.24	2.20 ± 0.22	1.97 ± 0.17	2.27 ± 0.17	2.33 ± 0.13	2.51 ± 0.21
1.15	2.04 ± 0.20	2.36 ± 0.23	1.91 ± 0.12	2.17 ± 0.17	2.12 ± 0.15	2.08 ± 0.11	2.37 ± 0.15
1.30	1.92 ± 0.19	2.26 ± 0.22	1.97 ± 0.13	1.86 ± 0.15	2.11 ± 0.16	2.03 ± 0.11	2.41 ± 0.17
1.45	1.76 ± 0.20	2.21 ± 0.25	1.92 ± 0.11	1.96 ± 0.17	1.90 ± 0.12	1.97 ± 0.09	2.28 ± 0.15
1.60	1.88 ± 0.23	2.26 ± 0.22	1.82 ± 0.09	1.83 ± 0.13	1.67 ± 0.11	1.95 ± 0.12	2.26 ± 0.19
1.75	1.62 ± 0.12	1.97 ± 0.17	1.60 ± 0.07	1.55 ± 0.11	1.54 ± 0.07	1.79 ± 0.08	2.19 ± 0.13
1.90	1.58 ± 0.13	1.82 ± 0.11	1.59 ± 0.07	1.39 ± 0.06	1.45 ± 0.08	1.73 ± 0.08	1.98 ± 0.10
2.05	1.70 ± 0.11	1.82 ± 0.12	1.63 ± 0.07	1.35 ± 0.05	1.47 ± 0.09	1.65 ± 0.09	1.90 ± 0.13
2.20	1.59 ± 0.12	1.82 ± 0.10	1.38 ± 0.07	1.39 ± 0.06	1.36 ± 0.09	1.46 ± 0.08	1.80 ± 0.09
2.35	1.45 ± 0.11	1.78 ± 0.09	1.38 ± 0.09	1.39 ± 0.06	1.18 ± 0.11	1.68 ± 0.11	1.83 ± 0.11
2.50	1.51 ± 0.10	1.80 ± 0.11	1.32 ± 0.08	1.38 ± 0.07	1.37 ± 0.10	1.63 ± 0.14	1.77 ± 0.13
2.65	1.40 ± 0.08	1.42 ± 0.08	1.41 ± 0.07	1.31 ± 0.08	1.30 ± 0.06	1.50 ± 0.10	1.77 ± 0.15
2.80	1.38 ± 0.07	1.53 ± 0.10	1.34 ± 0.08	1.30 ± 0.07	1.30 ± 0.06	1.32 ± 0.10	1.38 ± 0.08
2.95	1.35 ± 0.08	1.53 ± 0.10	1.28 ± 0.07	1.24 ± 0.07	1.09 ± 0.05	1.42 ± 0.11	1.55 ± 0.13

relatively strong, but relatively weak for currencies.

In order to illustrate this observation, the values of α are plotted in Fig 3 against the threshold δ for DEM and SPX time series.

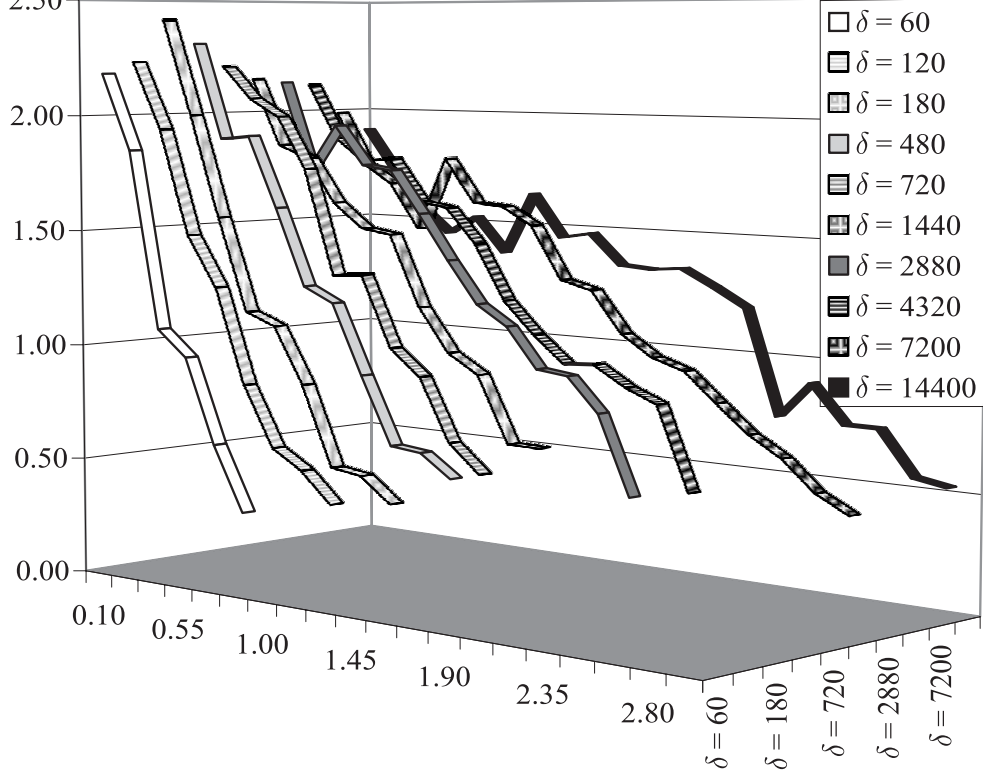
Fig. 3. α values for DEM and SPX under thresholds $\delta = 0.10\% \dots 2.95\%$ and $\tau=10$ days.



Next we study the dependence of α on the window width τ using the high frequency data of EUR-USD exchange rate. As mentioned above, low amount of statistics becomes an issue, when the threshold parameter is too high for

the given window width. Therefore only those values of α are plotted in Fig 4, where $R^2 > 0.7$.

Fig. 4. Scaling exponent α for EUR time series with different window widths τ under thresholds $\delta = 0.10\% \dots 2.95\%$.

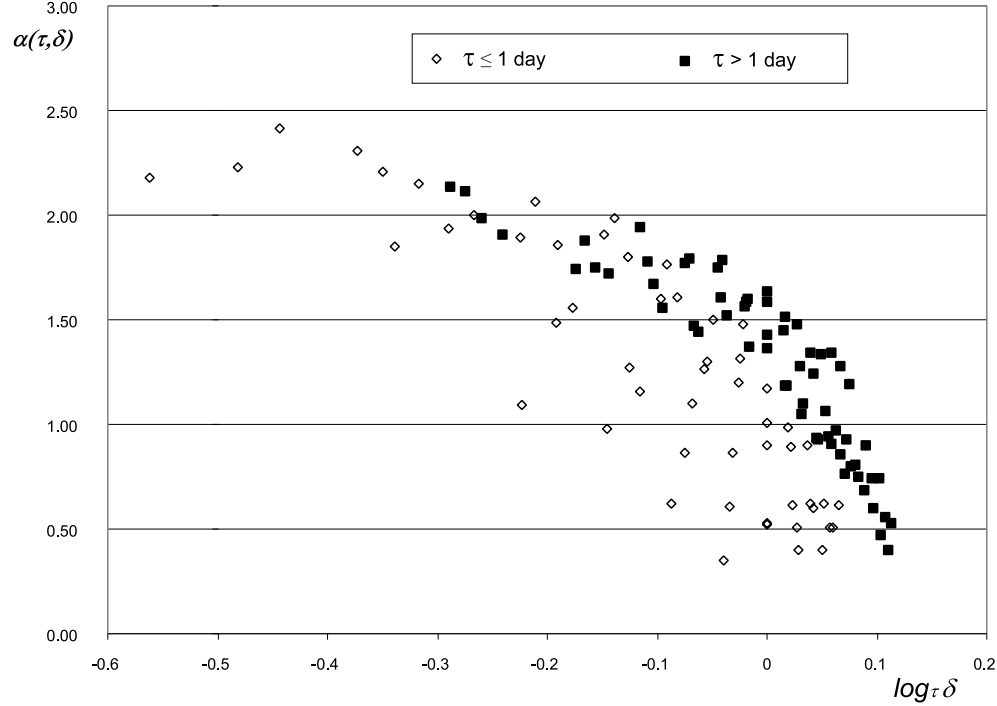


3.3 Discussions of the results

According to the theoretical expectations, α should be a decreasing function of δ , owing to Eq. 6 and to the circumstance that for $h > h_0$ (i.e. for $\alpha \leq 1$), $f(h)$ is a decreasing function of h . This trend is expected to hold for the range $\alpha > 1$, as well, due to simple reasoning. Increasing the threshold parameter δ leads to some fluctuations ceasing to be qualified as “large”, so that longer low-variability periods will emerge. Increasing the number of long periods, in its turn, corresponds to decreasing the scaling exponent α , in accordance with the results depicted in Fig 3.

In order to study, how well the theoretically derived relationship Eq. 6 (or Eq. 8) holds for the financial time series, the scaling exponent $\alpha(\delta, \tau)$ has been plotted versus $\log_\tau \delta$ in Fig. 5 (using the same data as in Fig. 4). Note that the data-points with $\tau > 2880 \text{ min} = 2 \text{ days}$ describe the scaling behaviour at the time scale above one day; the data-points in the range $\tau \leq 1 \text{ day}$ are influenced significantly by the intra-day dynamics of the exchange rate.

Fig. 5. Scaling exponent α for EUR time series with different values of τ and δ are plotted versus $\log_{\tau} \delta$, providing a data collapse for $\tau > 1$ day, $\alpha < 1$.



For the range of validity of Eq. 6 ($\alpha < 1$), the data-points with $\tau \geq 2$ lay reasonably well into a single curve, confirming the validity of Eq. 6. The data of Fig. 5 indicate also that at the time-scale around one day, there is a cross-over between different scaling regimes: the data with $\tau \leq 1$ day incorporate a mixed dynamics (which cannot be described by mafBm model) and lay no longer on the same curve. At the range $\alpha > 1$, there is no evident reason for the data to lay on a curve: all the corresponding data points are scattered in a form of a disperse cloud.

4 Conclusion

In this paper a new method for time series analysis is derived and analytically motivated. It is shown that for multi-affine time series, the scaling properties of low-variability periods are described by scaling exponent α as a function of threshold parameter δ and averaging window width τ . The scaling analysis of low variability periods offers a simple alternative to the multi-affine analysis of the time series, providing wider applicability range and somewhat higher time resolution. An open question is about the existence of a class of scale-invariant time series, more generic than multi-affine ones, violating Eq. 6.

Particular emphasize is paid to the financial time series. The analysis showed a good scaling behaviour of the currency exchange rate and stock index data and

confirmed the theoretical expectations, particularly the relationship between the scaling exponent α and the model parameters δ and τ (c.f. Eq. 6). The analysis of high-frequency data of EUR-USD exchange rate indicated that there is a cross-over between different scaling regimes at the time scale of one day. It was also observed that scaling exponent values for equity time series tend to be more sensitive with respect to the threshold parameter than currency time series.

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